Rank Of Matrix

A matrix is said to be of rank r, when

i) There is at least one minor of A of order r which does not vanish

ii) every minor of A of order (r+1) or higher vanishes

Rank = NO. of non-zero row in upper triangular martix

Briefly, the rank of matrix is the largest order of any non-vanishing minor of the matrix

two useful results

i) If a matrix has a non zero minor of order r, its rank is $\geq r$

ii) If all minors of a matrix of order $r+1$ are zero, its rank is $\leq r$

⇒ The rank of matrix A shall be denoted by $\rho(A)$

⇒ Invariance of Rank through elementary transformation

- Elementary Transformation do not alter the rank of a matrix i.e equivalent matrices have the same rank

The rank of the transpose of a matrix is the same as than of the original

matrix
The rank of a product of two matrices cannot be exceed the rank of either matrix i.e $\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$

⇒ Echelon form of Matrix
A matrix $A = (a_{ij})_{m \times n}$ is said to be in echelon form, if
i) every row of A which has all its entries 0 occurs below every row which has a non zero entry
ii) the number of zero preceding the first non zero element in a row is less than that the number of such zeros in the succeeding (or next) row.
iii) the first non-zero element in every row is unity
when a matrix is converted in echelon form, then the number of non-zero rows of the matrix is known as the rank of the matrix A

ex:- $A = \begin{bmatrix} 1 & 3 & -2 & 6 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in the echleon matrix

⇒ Normal form of a Matrix
Every non-zero matrix [Says $A = (a_{ij})_{m \times n}$] of rank r, by a sequence of elementary transformation can be reduced to the form $\begin{bmatrix} I_r & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$ $\begin{bmatrix} I_r \\ \cdots \\ 0 \end{bmatrix}$ $[I_r \vdots 0]$ or $[I_r$

where I_r is $r \times r$ unit matrix of order r and 0 represented null matrix of any order.
These form are said to be the normal form of canonical form of the given matrix A

Ques1) find the rank of matrix
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Sol:- let the matrix be A
$|A| = 1(20-12) - 2(5-4) + 3(6-8)$
$\Rightarrow |A| = 8 - 2 - 6 \Rightarrow |A| = 0$
so $\rho(A) < 3$
checking the matrix by taking any 2×2 minor.
suppose we take $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$ Here minor is non zero
and determinent is also non zero

So $\rho(A) = 2$
so here the rank of matrix is 2 //

⇒ Solving the ques by transmission
In this we mak R_2 and R_3 '0'

Step 1 $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

Step 2 :- we'll make this '0'

$\Rightarrow R_3 = R_3 - R_2$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ → Making this '0' in 3rd step

$\rho(A) = 2$

Step 3 - Here we'll make C_2 & C_3 '0'

$C_2 = C_2 - 2C_1$, $C_3 = C_3 - 3C_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ → Making this '0' in 4th step

Step 4 :- $C_3 \to 2C_3 + C_2$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Now this is Normal form

Ques 2) find one non-zero minor of highest order of the matrix $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ and hence find the rank of matrix A

Sol^n Solving by transmission form

$A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$

Making them 0

$\Rightarrow R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$\begin{bmatrix} -1 & -2 & 3 \\ 0 & 8 & -7 \\ 0 & 4 & 4 \end{bmatrix}$

we'll make this '0'

$\Rightarrow R_3 \to 2R_3 - R_2$

$\Rightarrow \begin{bmatrix} -1 & -2 & 3 \\ 0 & 8 & -7 \\ 0 & 0 & 15 \end{bmatrix}$ Here its rank is 3

$\rho(A) = 3$

Becaus last row is not fully 0

Nullity = Jitni rows 0 banegi utni nullity hogi

Q1) find the rank and nullity of

$A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$

Sol^n

$$\begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

Making them '0'

$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

Making them '0'

$\Rightarrow R_4 \to R_4 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\because$ Non 0 lines =

$\because$ Rank = 2

Nullity = 2

n(order of matrix)

$\Rightarrow \rho(A) + n(A) = n$

$\Rightarrow 2 + 2 = 4$

$\Rightarrow 4 = 4$ Hence Proved

Q2) find the rank of matrix

$$A = \begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$$

Sol^n Its determinent i.e $|A| = 0$

Now taking minor of 3X3 of it and if any minor is non 0 then its rank will be 3

$$M = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{bmatrix}$$

$|M| = 1(225 - 256) - 4(100 - 144) + 9(64 - 81)$

$\Rightarrow |M| = -31 + 176 - 153 = 176 + 180$

$\Rightarrow |M| = 357$

$\rho(A) = 3$ ans

Ques3) find the rank and normal form nullity of the following matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Sol^n
$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \rightleftharpoons R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 18 & 17 \end{bmatrix}$$

$R_3 \to 5R_3 - 4R_2$, $R_4 \to 5R_4 - 9R_2$

$$\Rightarrow A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\Rightarrow A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 3$

for normal form

$C_2 \to C_2 + C_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 2C_1$

$C_4 \to C_4 + 4C_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 5C_3 - 3C_2$

$C_4 \to 5C_4 - 7C_2$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 165 & 110 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to 3C_4 - 2C_3$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 165 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2/5$

$C_3 \to C_3/165$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the Normal form

Ques 4) Reduce the following matrix into normal form and hence find its rank.

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Sol^n $R_3 \rightleftarrows R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -8 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & 0 & -4 & -6 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 - C_1 \\ C_2 \to C_2 - C_1 \\ C_4 \to C_4 - 2C_1 \end{matrix}$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & 0 & -4 & -6 \end{bmatrix} \quad C_4 \to C_4 - 2C_3$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & 0 & -4 & 2 \end{bmatrix} \quad C_3 \to 3C_3 - C_2$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -12 & 2 \end{bmatrix} \quad C_3 \to C_3 + 6C_4$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2/-6 \\ C_4 \to C_4/2 \end{matrix}$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho(A) = 3$$

This is the Normal form

Application of matrix for solving system of linear equation

$AX = B$

- Homogenous ($B = 0$)
  - Always consistent
    - Zero solⁿ: $\rho(A) =$ No. of unknown, $|A| \neq 0$
    - Non zero solⁿ: $\rho(A) <$ No. of unknown, $|A| = 0$
- Non-Homogenous ($B \neq 0$)
  - Inconsistent: $\rho(A) \neq \rho(A:B)$ (No solⁿ)
  - Consistent: $\rho(A) = \rho(A:B)$
    - Unique solⁿ: $\rho(A) = \rho(A:B) =$ No. of unknowns
    - Infinite solⁿ: $\rho(A) = \rho(A:B) <$ No. of unknown (Remaining var are given arb value)

1) $x + y + z = -3$
$3x + y - 2z = -2$
$2x + 4y + 7z = 7$

Sol^n

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\Rightarrow [A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\Rightarrow [A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

$\Rightarrow \rho(A:B) = 3$, $\rho(A) = 2$
$\Rightarrow \rho(A:B) \neq \rho(A)$ No sol^n

2) $x + y + z = 6$
$x - y + z = 2$
$2x + 4y - z = 1$

Sol^n

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 4 & -1 & 1 \end{array}\right] \quad [A:B]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & -2 & -3 & -11 \end{array}\right]$$

$R_3 \to 2R_3 - R_2$

$$\Rightarrow [A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & -6 & -18 \end{array}\right]$$

$\rho(A:B) = 3 = \rho(A) =$ No. of unknowns
Unique sol^n

$\Rightarrow x + y + z = 6$
$-2y = -4$
$-6z = -18$

$\Rightarrow z = 3$, $y = 2$, $x = 1$

3) $x_1 - x_2 + x_3 = 2$
$3x_1 - x_2 + 2x_3 = -6$
$3x_1 + x_2 + x_3 = -18$

Sol^n $$[A:B] = \left[\begin{array}{ccc:c} 1 & -1 & 1 & 2 \\ 3 & -1 & 2 & -6 \\ 3 & 1 & 1 & -18 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 4 & -2 & -24 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A:B) = 2 = \rho(A:B) \neq$ No. of unknowns

Infinite sol^n

$x_1 - x_2 + x_3 = 2$

$2x_2 - x_3 = -12$

$x_3 = k$

$\Rightarrow 2x_2 - k = -12$

$\Rightarrow \boxed{x_2 = \frac{k-12}{2}}$

$\Rightarrow x_1 - \left[\frac{k-12}{2}\right] - k = 2$

$\Rightarrow x_1 - k + 12 - 2k = 4$

$\Rightarrow x_1 = 4 - 3k - 12 \Rightarrow x_1 = -8 - 3k$

4) $x + 2y + 3z = 0$

$3x + 4y + 4z = 0$

$7x + 10y + 12z = 0$

Sol^n

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 3 =$ No. of unknowns

Zero sol^n i.e $x = y = z = 0$

5) $x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

Sol^n

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 \neq$ No. of unknown

Non zero sol^n

$x + 3y - 2z = 0$

$-7y + 8z = 0$

$z = k$

$7y = 8k \Rightarrow y = 8k/7$

$\Rightarrow x + \frac{24k}{7} - 2k = 0$

UNIT- 3

Groups

G_1 = closure property

$a, b \in G \Rightarrow a*b \in G \quad \forall a, b \in G$

G_2 = Associative property

$a*(b*c) = (a*b)*c \quad \forall a, b \in G$

G_3 = Existence of Identity

$\exists\ e \in G$ such that $a*e = a = e*a$ $\forall a \in G$

G_4 = Existence of Inverse

$\forall a \in G\ \exists\ a^{-1} \in G$ such that $a*a^{-1} = a^{-1}*a = a$

Abelian group

G_5 = Commutative property

$a*b = b*a \quad \forall a, b \in G$

Q) Show that set of cube roots of unity forms an abelian group w.r.t $(\cdot)$

Sol^n $X = 1^{1/3} \Rightarrow x^3 - 1 = 0$

$\Rightarrow (x-1)(x^2+x+1) = 0$

$\Rightarrow x = 1, \ x = \dfrac{-1 \pm i\sqrt{3}}{2}$

$$\omega = \frac{-1 + \sqrt{3}\, i}{2}$$

$$\omega^2 = \frac{-1 - \sqrt{3}\, i}{2}$$

$G(1, \omega, \omega^2)$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$\because \omega^3 = 1$

$\omega^4 = \omega$

① G is closure as all entities are element of G

② G is associative :-

$1.(\omega.\omega^2) = (1.\omega).\omega^2$

$1.1 = \omega.\omega^2$

$1 = 1$ HP

③ Coloumn 1 is same as Head coloumn then $e = 1 \in G$

④ for inverse:-

a	a^{-1}
1	1
ω	ω^2
ω^2	ω

⑤ Matrix is symmetric along the diagonal so the Matrix is commutative Hence it is an Abelian group

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Q) $G(0,1,2,3)$, $G(+_4)$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

① It is closure as all the entities belongs to G

② for associative :-

$\Rightarrow 1 +_4 (2 +_4 3) = (1 +_4 2) +_4 3$

$\Rightarrow 1 +_4 (1) = 3 +_4 3$

$\Rightarrow 2 = 2$ HP

③ Identity element $= 0$

④ Inverse :-

a	a^{-1}
0	0
1	3
2	2
3	1

5 All the diagonal element are same
Hence it is abelian

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Q) Let $G = Q - \{-1\}$, $a*b = a+b+ab$ then PT $(G*)$ is an abelian Group

Sol^n for closure:-

$a, b \in G = Q - \{-1\}$

$a*b = a+b+ab \in G$

let $a+b+ab = -1$

$\Rightarrow a+b+ab+1 = 0$

$\Rightarrow a(1+b)+1(1+b) = 0$

$\Rightarrow (1+b)(1+a) = 0$

$\Rightarrow a = -1$ or $b = -1$ this is contradiction

so $a*b = a+b+ab \neq -1$

$\therefore a*b \in Q \ \forall a, b \in G$

$G_2 =$ for associative:-

$a*(b*c) = (a*b)*c$

LHS:- $a*[b+c+bc] \Rightarrow a+[b+c+bc]+a[b+c+bc]$

$\Rightarrow a+b+c+bc+ab+ac+abc$

$\Rightarrow$ RHS:- $(a*b)*c$

$\Rightarrow [a+b+ab]*c \Rightarrow [a+b+ab]+c+[a+b+ab]*c$

$\Rightarrow a+b+c+ab+ac+bc+abc$

Hence LHS = RHS HP

$G_3 =$ Identity:- $a*e = a$

$a+e+ae = a$

$e(1+a) = 0$

$\Rightarrow e = 0 \in G$

G_4 :- Inverse :- $e=0$
$a \in G$ let $b = a^{-1}$
$a * b = e \Rightarrow a+b+ab=0$
$\Rightarrow a+b(1+a)=0$
$\Rightarrow b = \dfrac{-a}{1+a} \in G$

$G_5 = a*b = b*a$
LHS $= a*b = a+b+ab$
RHS $= b*a = b+a+ba$
LHS = RHS HP

Subgroup = A non empty subset H of a group G is called a subgroup of G if H is a group w.r.t composition in G
operation like $(\cdot)(+)$ etc
H is a subgroup of G if and only if $ab^{-1} \in H$ $\forall a,b \in H$ where b^{-1} is the inverse of b in G

$\Rightarrow$ Theoram 1:- If H_1 & H_2 are two subgroups of a group G then P.T $H_1 \cap H_2$ is also a subgroup of G
OR
PT Instersection of two subgroups of a Group G is a subgroup of G

Q) Let $a,b \in H_1 \cap H_2$ To prove $ab^{-1} \in H_1 \cap H_2$
Proof - $\because a,b \in H_1 \cap H_2$

$a,b \in H_1$ $a,b \in H_2$
$\Rightarrow ab^{-1} \in H_1$ $ab^{-1} \in H_2$

$ab^{-1} \in H_1 \cap H_2$ HP

$\Rightarrow$ Theoram 2:- Union of two sub is not necesserality a subgroup

Q) $(I+)$ is a group
$H_1 = \{0, \pm 2, \pm 4 \ldots\}$
$H_2 = \{0, \pm 3, \pm 6 \ldots\}$
H_1 is a sub group
H_2 is a sub group

$H_1 \cup H_2 \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \ldots\}$
But the union of H_1 & H_2 is not the subgroup

Cyclic group:- A group G is a cyclic group if there exist an element $a \in G$ such that $G = [a]$
i.e every element of G can be expressed as some integral power of a
a is called the generator of G

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$G = \{ \ldots\ldots a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3 \ldots \}$

If the opposition of the group G is addition $(+)$ then

$G = [a] = \{ \ldots -3a, -2a, -a, 0, a, a, 2a, \ldots \}$

Q1) Prove that the set $G = \{x \mid x^n = 1\}$ of n^{th} root of unity is a finite multiplicative cyclic group of order n

Sol^n Let $x_1, x_2 \in G$ then $x_1^n = 1$ & $x_2^n = 1$

But $x_1^n = 1$, $x_2^n = 1 \Rightarrow x_1^n x_2^n = 1 \cdot 1 = 1$

$\Rightarrow (x_1 x_2)^n = 1 \quad \Rightarrow x_1 x_2 \in G$

$\therefore x_1 \in G, x_2 \in G \quad \Rightarrow x_1 x_2 \in G$

So G is closed for multiplication

Also $1^n = 1 \Rightarrow 1 \in G$ which is the identity for multiplication further for each $x \in G$

$x \in G = \quad x^n = 1 \Rightarrow \dfrac{1}{x^n} = 1 \quad \Rightarrow \left(\dfrac{1}{x}\right)^n = 1$

$\Rightarrow \dfrac{1}{x} \in G$

So each element of G is invertible lastly the multiplication of number is associative, so it is also associative in G

Hence G is a group

$x^n = 1$ has exactly n roots

so $O(G) = n$

Checking for cyclic group

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$x^n = 1 \quad \Rightarrow \quad x = (1)^{1/n}$

$\Rightarrow x = [\cos 2m\pi + i \sin 2m\pi]^{1/n}$

$\{\because \cos 2m\pi = 1, \ \sin 2m\pi = 0\}$

$\Rightarrow x = \cos \dfrac{2m\pi}{n} + i \sin \dfrac{2m\pi}{n}$

$\Rightarrow x = e^{2 \sin m\pi / n}$

$e^{2 \sin m\pi/n}$ is generator

Q2) Find all the generators of the cyclic group $(G = \{0, 1, 2, 3, 4, 5\}, +_6)$

Sol^n $1(0) = 0 \quad \Rightarrow O(0) = 1$

$1(1) = 1 = 1 \qquad 4(1) = 1+1+1+1 = 4$

$2(1) = 1+1 = 2 \qquad 5(1) = 1+1+1+1+1 = 5$

$3(1) = 1+1+1 = 3 \qquad 6(1) = 1+1+1+1+1+1 = 0$

$\Rightarrow O(1) = 6$

$1(2) = 2$

$2(2) = 2+2 = 4$

$3(2) = 2+2+2 = 6 = 0 \qquad \Rightarrow O(2) = 3$

$1(3) = 3$

$2(3) = 3+3 = 6 = 0 \qquad \Rightarrow O(3) = 2$

$1(4) = 4 \qquad 3(4) = 4+4+4 = 12 = 0$

$2(4) = 4+4 = 8 = 2$

$\Rightarrow O(4) = 3$

$1(5) = 5 \qquad 4(5) = 5+5+5+5 = 20 = 2$

$2(5) = 5+5 = 10 = 4$

$3(5) = 5+5+5 = 15 = 3 \qquad 5(5) = 5+5+5+5+5 = 25 = 1$

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$6(5) = 5+5+5+5+5+5 = 30 = 0$

$O(5) = 6$

observing the order of all element of G we find

$O(1) = O(5) = 6 = O(G)$

Therefore $G = [1] = [5]$ i.e 1 and 5 are two generator of G

Q3) Find all the generator of the cyclic group ($G = \{1,2,3,4\}, \times_5$)

Soln $O(H) = 4$

Its generator is that element whose order is 4

$1^2 = 1 \Rightarrow O(1) = 1$

$2^1 = 2, 2^2 = 4, 2^3 = 3, 2^4 = 1$

$O(2) = 4$

$3^1 = 3, 3^2 = 4, 3^3 = 2, 3^4 = 1$

$O(3) = 4$

$4^1 = 4, 4^2 = 1 \quad O(4) = 2$

Clearly there are 2 generator i.e 2 and 3

$O(2) = 4 = O(3) = O(G)$

Q4) The group $[Z = \{0,1,2,3\}, +_4]$ is a cyclic group with 1 and 3 as its two generator because

$Z = \{1(1) = 1, 2(1) = 2, 3(1) = 3, 4(1) = 0\} = [1]$ and $Z_4 = \{1(3) = 3, 2(3) = 2, 3(3) = 1,$

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$4(3) = 0\} = [3]$

Soln $1(0) = 0 \quad O(0) = 1$

$1(1) = 1$

$2(1) = 1+1 = 2$

$3(1) = 1+1+1 = 3$

$4(1) = 1+1+1+1 = 0 \quad O(1) = 4$

$1(2) = 2$

$2(2) = 2+2 = 0 \quad O(2) = 2$

$1(3) = 3$

$2(3) = 3+3 = 6 = 2$

$3(3) = 3+3+3 = 9 = 1$

$4(3) = 3+3+3+3 = 12 = 0 \quad O(3) = 4$

Hence 1 and 3 are the generator because $O(G) = 4 = O(1) = O(3)$

THEORAMS

$\Rightarrow$ Theoream 1 = Every cyclic group is an abelian

Proof: Let $G = [a]$ be a cyclic group and $x, y \in G$ where

$x = a^m, y = a^n \quad m, n \in Z$

then $xy = a^m a^n = a^{m+n} = a^{n+m}$

$= a^{nm} = yx$

$\Rightarrow xy = yx$

$\therefore$ G is an abelian group

Remark: An abelian group need not to

to be cyclic
ex= (ℓ+) is an abelian but not cyclic

⇒ Theoram 2:- If a is a generator of a cyclic Group G, then a^{-1} is also its generator.

Proof: Let G = [a] be a cyclic group
$x \in G$

Since G is a cyclic group, so there exist an integer m such that

$x = a^m \Rightarrow x = (a^{-1})^{m}$ $[-m \in z]$

⇒ x can also be expressed as some integral power of a^{-1}

⇒ a^{-1} is also the generator of G

Therefore $G = [a] \Rightarrow G = [a^{-1}]$

⇒ Theoram 3:- The order of a finite cyclic group is equal to the order of its generator i.e

O(finite cyclic Group) = O(generator of group)

Proof: G = [a] be a finite cyclic group and O(a) = n

Let $H = \{a, a^2, a^3, \ldots a^n = e\}$

clearly H is a sub group of G whose order is n

Case 1:- When $m \le n$: If $a^m \in G$ then $a^m \in H$
$\therefore H < G$ — ①

Case 2:- $m > n$, $m = qn + r$ $0 \le r < n$, $q, r \in z$

$\Rightarrow a^m = a^{qn+r} = (a^n)^q . a^r = e a^r$

$\Rightarrow G \subset H$ — ②

from eq^n ① & ②

$\Rightarrow G = H$

But O(H) = n

O(G) = n = O(a)

⇒ Cor- A ~~fou~~ finite group of order n is cyclic iff it has an element of order n

Proof:- Let G = [a] be a finite cyclic group of order n

Then by the above theoram an element a exist in G such that

O(a) = O(G) = n

Conversely:- Let G be a finite cyclic group of order n in which an element a exist such that

O(a) = n Now if H = [a] then

H ⊂ G and by the above theorams

O(a) = n ⇒ O(H) = n

O(H) = O(G)

Similarly G is finite group such that

H <G and O(G) = O(H)

⇒ G = H = [a]

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$\Rightarrow G$ is a cyclic group generated by a

$\Rightarrow$ Theoram 4:- Every infinite cyclic group has two and only two generators

Proof:- Let $G = [a]$ be an infinite cyclic group. Then by theoram 2 a^{-1} is also a generator of G

To show:- $a^{1} \neq a^{-1}$

Let $a = a^{-1}$, then $a = a^{-1}$

$\Rightarrow aa = a^{-1}a \Rightarrow a^2 = e$

$\Rightarrow O(a) = 2 \Rightarrow O(G) = 2$

which is not possible because G is an infinite group. Therefor $a \neq a^{-1}$

To show that G does not have any generator other than these two

Let, if possible, a^m, $m \neq \pm 1$ be also a generator of G.

The for $a \in G$ there exist an integer n such that

$a = (a^m)^n = a^{mn}$ [Multiply by a^{-1}]

$\Rightarrow aa^{-1} = a^{mn}a^{-1}$

$\Rightarrow e = a^{mn-1}$

$\Rightarrow O(a)$ is finite

$\Rightarrow O(G)$ is finite

which contradict that G is infinite

Hence a^m can not a generator of G

$m = 1$ or -1 Consequently G has exactly two generator a and a^{-1}

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$\Rightarrow$ Theoram 5:- Every subgroup of a cyclic group is also cyclic

Proof:- Let $G = [a]$ be a cyclic group and H be a sub group of G

if $H = G$ or $H = \{e\}$, then clearly H is also cyclic

if H is a proper sub group of G, then H contains at least one element a^m ($m \in Z, m \neq 0$) other than the identity

$a^m \in H \Rightarrow a^{-m} \in H$ [$\because H$ is a subgroup]

Since $m \neq 0$, therefore $m > 0$ or $-m > 0$

$\Rightarrow$ There exist positive integral power of a in H

Let m be the least positive integer such that $a^m \in H$

To prove:- $H = [a^m]$

Let $a^n \in H$, then by division algorithm, there exist two integers q and r such that

$n = mq + r \quad 0 \leq r < m$

or $n - mq = r$

Now since $a^m \in H \Rightarrow (a^m)^q = a^{mq} \in H$

$\Rightarrow (a^{mq})^{-1} = a^{-mq} \in H$

$\therefore a^n \in H$, $a^{-mq} \in H \Rightarrow a^n . a^{-mq} = a^{n-mq} \in H$

But m is the least +ve integer such that $a^m \in H$ and $0 \leq r < m$

Therefore $r = 0$ Consequently $n = mq$

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and so $a^n = a^{mq} = (a^m)^q \Rightarrow H = [a^m]$

Therefore every subgroup of G is cyclic

COSETS

① let $(G, *)$ be a group & H are its subgroup let $a \in G$, then

$Ha = \{ha : h \in H\}$

is called the right coset of H in G

Similarly

$aH = \{ah : h \in H\}$ is called left coset of H in G

② If G is abelian then $Ha = aH$

Let $(G, +)$ be a group then

$H + a = \{h + a : h \in H\}$

$a + H = \{a + h : h \in H\}$

ex:- $G\{1, -1, i, -i\}$

let $H = \{1, -1\}$

$1 \in G$

$H \cdot 1 = \{1, -1\}$

$-1 \in G$

$H \cdot (-1) = \{-1, 1\}$

$i \in G$

$H \cdot i = \{i, -i\}$

$(-i) \in G$

$H \cdot (-i) = \{-i, i\}$

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Here $H \cdot 1 = H \cdot (-1)$ & $H \cdot i = H \cdot (-i)$

$H1 \cap Hi = \phi$

Coset decomposition iska mtlb ki agar hum jitne bhi disjoint hai unka union kare to vo group hi ban jata hai

$G(1, -1, i, -i) = H1 \cup Hi$

$\Rightarrow \boxed{He = H = eH}$

$\Rightarrow a \in H \subset G$ then $\boxed{Ha = H = aH}$

ex:- $I = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

$H = \{0, \pm 2, \pm 4 \ldots\}$

$H + 0 \{\ldots -4, -2, 0, 2, 4, \ldots\}$

$H + 1 \{\ldots -3, -1, 1, 3, 5, \ldots\}$

$H + (-1) \{\ldots -5, -3, -1, 1, 3, \ldots\}$

$H + (2) \{\ldots, -2, 0, 2, 4, 6, \ldots\}$

$H + (-2) \{\ldots, -6, -4, -2, 0, 2, \ldots\}$

Q) $G = \{a, a^2, a^3, a^4 = e\}$ $(G, \cdot)$, $H\{e, a^2\}$

Sol $a \in G$

$H \cdot a = \{ae, a^3\} \Rightarrow H \cdot a = \{a, a^3\}$

$a^2 \in G$

$H \cdot a^2 \{a^2 e, a^4\} \Rightarrow H \cdot a^2 = \{a^2, e\}$

$a^3 \in G$

$H \cdot a^3 \{a^3 e, a^5\} \Rightarrow H \cdot a^3 = \{a^3, a\}$

$e = a^4 \in G$

$H \cdot a^4 \{a^4 e, a^2 \cdot a^4\} \Rightarrow H \cdot a^4 \{e^2, a^2\}$

THEORAM 1:- If H is any subgroup of G and $h \in H$, then $Hh = H = hH$

(1) Let $x \in Hh$

$x = h_1 h \; : h_1 \in H$

$x = h_1 h \in H$ ($\because H$ is closed)

$\therefore Hh \subseteq H$

(2) $y \in H$

$y = y \cdot e \Rightarrow y = y(h^{-1}h)$

$\Rightarrow y = (yh^{-1})h \Rightarrow y = h_1 h, \; h_1 \in H$

$\Rightarrow y = h_1 h \in H \quad \therefore H \subseteq Hh$

$\Rightarrow Hh = H$

THEORAM 2:- If H is a subgroup of Group G, then $Ha = Hb \Leftrightarrow ab^{-1} \in H \; \forall a, b \in G$

Proof :- $Ha = Hb \Leftrightarrow ab^{-1} \in H$

(1) $Ha = Hb \Rightarrow ab^{-1} \in H$ (2) $ab^{-1} \in H \Rightarrow Ha = Hb$

(1) $Ha = Hb \Rightarrow ab^{-1} \in H$

$\because a \in Ha$

$a \in Hb$ multiply by b^{-1}

$ab^{-1} \in Hbb^{-1}$

$\Rightarrow ab^{-1} \in He \Rightarrow ab^{-1} \in H$

(2) $ab^{-1} \in H \Rightarrow Ha = Hb$

$\because ab^{-1} \in H$

$Hab^{-1} = H$ multiply by b

$Hab^{-1}b = Hb$

$Hae = Hb \Rightarrow Ha = Hb$

THEORAM 3:- Any two right coset of subgroup are either disjoint or identical

Proof:- Let $Ha \cap Hb = \phi$

Let $c \in Ha \cap Hb$

$\Rightarrow c \in Ha$ & $c \in Hb$

$c = h_1 a$ & $c = h_2 b$

$h_1 \in H, h_2 \in H$

$h_1 a = h_2 b$ multiply by h_1^{-1}

$h_1 h_1^{-1} a = h_2 h_1^{-1} b$

$ae = h_2 h_1^{-1} b$

$a = h_2 h_1^{-1} b$

$Ha = Hh_1^{-1} h_2 b \quad \{\because h_1^{-1} h_2 \in H\}$

$Ha = Hb$

THEORAM 4:- If H is a subgroup of a group G and $a \in G$ then: $a \in aH$ and $a \in Ha$

Proof:-

Let e be the identity element of G so also of H. Then for every $a \in G$, $e \in H$

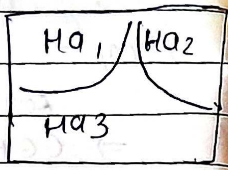
$\Rightarrow ae = a \in aH$

and $e \in H \Rightarrow ea = a \in Ha$

Remark:- from the above theoram it is clear that $aH \neq \phi$ and $Ha \neq \phi$ $a \in G$

THEORAM 5 :- Lagrange's theoram

The order of every subgroup of a finite group is a divisor of the order of group



Proof:- $o(G) = n$

$o(H) = m \,,\, m \leq n$

Let $a \in G$

$Ha = \{ha : h \in H\}$

$o(Ha) = o(H) = m$

$G = Ha_1 \cup Ha_2 \cup \dots \cup Ha_k$

$n(G) = n(Ha_1) + n(Ha_2) + \dots n(Ha_k)$

$n = m + m + m + \dots m$ k times

$n = mk$

$\frac{n}{m} = k$ $\quad k \in$ +ve integer

Hence proved

The converse of Lagrange's theoram is not always true i.e if m is a divisor of $n = o(G)$, then it is not necessary that G has a subgroup of order m

Normal subgroup:- A group H of a group G is said to be a normal subgroup of G if $\forall x \in G$ & $\forall h \in H$

$\boxed{-i \cdot i \cdot (-1) = -1 \in H}$ Normal subgroup

$(1, -1)$ is Normal subgroup of fourth root of unity

$\Rightarrow$ H is a normal subgroup of G if and only if $xHx^{-1} \in H \; \forall x \in G$

THEORAM :- A subgroup H of group G is a normal subgroup of G if & only if each left coset of H in G is equal to a right coset of H in G

$xH = Hx \quad \forall x \in G.$

$xH = Hx \; \forall x \in G$

$\iff$ H is normal

Proof:- H is normal	$xH = Hx$
$xHx^{-1} = H$	$xHx^{-1} = Hxx^{-1}$
$(xHx^{-1}).x = Hx$	$xHx^{-1} = He$
$xHx^{-1}x = Hx$	$xHx^{-1} = H$
$xHe = Hx$	$\Rightarrow$ H is normal
$\boxed{xH = Hx}$	
Hence prove	

Ques) prove that intersection of two normal subgroup of a group G is also a normal subgroup of G

$\forall x \in G, \; h \in H_1 \cap H_2$

To prove:- $xhx^{-1} \in H_1 \cap H_2$

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$h \in H_1$ & $h \in H_2$

$x \in G, h \in H_1$ $\quad$ $x \in G, h \in H_2$

$xhx^{-1} \in H_1$ $\quad$ $xhx^{-1} \in H_2$ {$\because H_1, H_2$ Normal}

$xhx^{-1} \in H_1 \cap H_2$

⇒ Proper and Improper Normal Subgroup
It can be observed that every group G has atleast following two normal subgroup
1 G itself
and 2. {e} the group consisting of the identity alone
These two subgroups are called Improper normal subgroup of G a normal subgroup other than these two is called a proper normal subgroup

Simple Group :- A group which has no proper normal subgroup is called a simple group

example :- every group of prime order is simple because such a group has no proper subgroup

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*THEORAM :- Every Subgroup of an abelian group is a normal Subgroup
Let H be a subgroup of any comutative group G
if $x \in G$ and $h \in H$ then

$\Rightarrow xhx^{-1} = (hx)x^{-1}$ [$\because$ G is commutative]

$\Rightarrow xhx^{-1} = h(xx^{-1})$ [by associativity]

$\Rightarrow xhx^{-1} = he \quad \Rightarrow xhx^{-1} = h \in H$

Thus, $x \in G, h \in H \Rightarrow xhx^{-1} \in H$

$\therefore$ H is normal subgroup of G

THEORAM :- A Subgroup H of a Group G, is a normal subgroup of G iff the product of two right (left) coset of H in G is again a right (left) coset of H in G

Proof :- (⇒) Let H is a normal subgroup of G
if $a, b \in G$ then Ha and Hb are two right coset of H in G
Let $h_1 a \in Ha$ and $h_2 b \in Hb$ then

$h_1 a h_2 b \in Ha\,Hb$

But $h_1 a h_2 b = h_1 (a h_2) b$

$\Rightarrow h_1 a h_2 b = h_1 (h_2 a) b$ [$\because Ha = aH \Rightarrow a h_2 = h_2 a, h_2 \in H$]

$\therefore HaHb \subset Hab$ — (1)

Again for every $h \in H$

$h(ab) = (ha)b = (ha)eb \in HaHb$

$Hab \subset HaHb$ — (2)

Hence ① and ② $\Rightarrow HaHb = Hab$

$H \trianglelefteq G \Leftrightarrow HaHb = Hab$

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Quotient Group

$\frac{G}{N} = \{Na : a \in G\}$ is a group w.r.t multiplication

A coset and their group is called a Quotient Group

$\Rightarrow$ collection of right coset of normal subgroup is called quotient subgroup

$\Rightarrow$ Multiplication of cosets

$Na \cdot Nb = Nab$

1 closure property :- $Na, Nb \in \frac{G}{N} \Rightarrow Na \cdot Nb \in \frac{G}{N}$

$a, b \in G$

$\Rightarrow ab \in G$

$Na \cdot Nb = Nab \in \frac{G}{N}$

2 Associative :-

To prove :- $Na \cdot (Nb \cdot Nc) = (Na \cdot Nb) \cdot Nc$

LHS $= Na \cdot (Nb \cdot Nc)$

$= Na \cdot (Nbc)$

$= Na(bc)$

RHS $= (Na \cdot Nb) \cdot Nc$

$= (Nab)Nc$

$= N(ab)c$

$\Rightarrow N(ab)c = Na(bc) \quad \{\because a, b, c \in G\}$

3. Identity :- $Ne = N \in \frac{G}{N}$

$Na \cdot (Ne) = Na$

$Na \cdot Ne = Nae$

$= Na$

4. Inverse :- $Na \in \frac{G}{N}$, $Na^{-1} \in \frac{G}{N}$

St $Na\, Na^{-1} = Naa^{-1} = Ne \Rightarrow N$

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THEORAM = If H_1 and H_2 are two normal subgroup of a group G, then prove that $G/H_1 = G/H_2 \Leftrightarrow H_1 = H_2$

Proof = ① $H_1 = H_2 = \frac{G}{H_1} = \frac{G}{H_2}$

$\because H_1 = H_2 \Rightarrow \frac{G}{H_1} = \frac{G}{H_2}$

② $\frac{G}{H_1} = \frac{G}{H_2} \Rightarrow H_1 = H_2$

$N \in \frac{G}{N}$

$\because H_1 \in \frac{G}{H_1} \Rightarrow H_1 \in \frac{G}{H_2}$

H_1 is a right coset of H_2 in G

$G/H_2 = \{H_2 a$, $a \in G$

$H_1 = H_2 a$

$e \in H_1$ & $e \in H_2$

$H_1 \cap H_2 \neq \phi \Rightarrow H_1 = H_2$

$\therefore$ Any two cosets of a subgroup are either disjoint or identical

Ques) If $G = \{a, a^2, a^3, a^4, a^5, a^6 = e\}$ is a group and $H = \{a^3, a^6 = e\}$ is it normal subgroup find G/H

Sol $\frac{G}{H} = \{Hb : b \in G\}$

$Ha = \{a^4, a\} = Ha^4\{a, a^4\}$

$Ha^2 = \{a^5, a^2\} = Ha^5\{a^2, a^5\}$

$Ha^3 = \{a^6 = e, a^3\} = Ha^6\{a^3, e\}$

$\frac{G}{H} = \{Ha, Ha^2, Ha^3\}$

$\Rightarrow G/H$ is a Quotient group

$\cdot$	Ha	Ha^2	$\boxed{Ha^3}$
Ha	Ha^2	$\boxed{Ha^3}$	$Ha^4 = Ha$
Ha^2	$\boxed{Ha^3}$	$Ha^4 = Ha$	$Ha^5 = Ha^2$
Ha^3	$Ha^4 = Ha$	$Ha^5 = Ha^2$	$\boxed{Ha^6 = Ha^3}$

$(Na)(Nb) = Nab$

1. Closure ✓
2. Associative ✓
3. Identity $= Ha^3$
4. Inverse :-

element	inverse
Ha	Ha^2
Ha^2	Ha
Ha^3	Ha^3

Ques) Find the Quotient group G/H and also prepare its operation table when:
$G = [\{1, -1, i, -i\}, x]$, $H = [\{1, -1\}, x]$

Soln

$H \cdot 1 = \{1, -1\} = H$

$H \cdot (-1) = \{-1, 1\} = H$

$H \cdot i = \{i, -i\} = Hi$

$H \cdot (-i) = \{-i, i\} = Hi$

$\frac{G}{H} = \{H, Hi\}$

$\Rightarrow G/H$ is a quotient group

$\cdot$	$\boxed{H}$	Hi
H	$\boxed{H}$	Hi
Hi	Hi	$\boxed{H}$

1. Closure ✓
2. Associative ✓
3. Identity $= H$ ✓
4. Inverse :-

element	inverse
H	H
Hi	Hi

THEORAM :- Every Quotient group of an abelian group is abelian but not converse

Proof :- Let H be a normal subgroup of an abelian group G and $a \in G, b \in G$ then

$a \in G, b \in G \Rightarrow Ha \in G/H, Hb \in G/H$

$\therefore HaHb = Hab$

$HaHb = Hba$ $\{\because G$ is commutative $\Rightarrow ab = ba$

$HaHb = Hb\,Ha$

Thus we see that

$Ha \in G/H, Hb \in G/H \Rightarrow HaHb = HbHa$

$\therefore G/H$ is also commutative

Conversely : The converse is not necessarily true.

For example :- S_3/A_3 is an abelian group while S_3 is a non abelian group. The order of group S_3/A_3 is 2 and every group of order 2 is an abelian

THEORAM :- Every quotient group of cyclic group is cyclic but not conversely

Proof :- Let H be a normal subgroup of any cyclic group $G = [a]$ since every element of G is of the form $a^n, n \in Z$

therefore let $Ha^n \in G/H$ Then

$Ha^n = H(a \cdot a \ldots n \text{ times})$

$\Rightarrow Ha^n = HaHa \ldots Ha (n \text{ times})$ $(\because Hab = HaHb)$

$\Rightarrow Ha^n = Ha^n$ $(\because G/H = [Ha])$

Therefore G/H is also a cyclic group whose generator is Ha

Conversely :- The converse is not necessarily true. For example :- S_3/A_3 being a group of order 2, necessarily cyclic but S_3 is not cyclic group

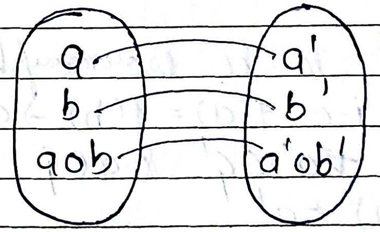
Homomorphism Of Groups

Let (G, o) and (G', o') be two groups. A mapping f from a group G to G' is said to homomorphism if $f(a\,o\,b) = f(a)\,o'\,f(b)$

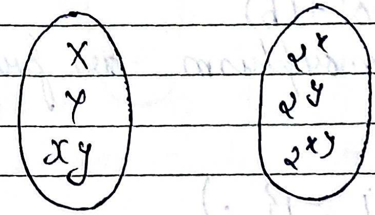
(G, o) (G, o')

$f(x) = 2^x$

$(G, \cdot)$ $(G', +)$



$f.\ X \to Y$



$f(x) = 3x$

$f(xy) = f(x) + f(y)$

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$f(x, y) = 2^{xy}$

$f(x) + f(y) = 2^x + 2^y$

$(G, +)$ $(G', \circ)$

G	G'
x	2^x
y	2^y
$x+y$	2^{x+y}

$f(x) = 3x$

$f(x+y) = f(x) \cdot f(y)$

$f(x+y) = 2^{x+y}$

$f(x) \cdot f(y) = 2^x \cdot 2^y = 2^{x+y}$

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Isomorphism of group

Let (G, o) and (G', o') be two group then

$f: G \to G'$ is said to be isomorphism if

1. f is one-one i.e $f(a) = f(b) \to a = b$
2. f is onto i.e $\forall a' \in G'$ & $a \in G$ such that $f(a) = a'$
3. f is homomorphism i.e $f(aob) = f(a) \, o' \, f(b)$

properties of homomorphism are property of isomorphism

ex:- let $G = (\{1, -1\}, \cdot)$ & $G' = (\{0, 1\}, +_2)$ then show that

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$G \cong G' \to$ is isomorphism

$\Rightarrow$ $(G, \cdot)$

$\cdot$	$\boxed{1}$	-1
1	1	-1
-1	-1	1

$(G', +_2)$

$+_2$	0	1
0	0	1
1	1	0

G	G'
$1 \to$	0
$-1 \to$	1

$f(1) = 0$

$f(-1) = 1$

$f(1 \cdot (-1)) = f(1) +_2 f(-1)$

$LHS = f(1 \cdot (-1)) = f(-1) = 1$ //

$RHS = f(1) +_2 f(-1) = 0 +_2 1 = 1$ //

Hence it is isomorphism

$\Rightarrow$ Let $G = (I, +)$ & $G' = (\{2^m : m \in I\}, \cdot)$ then show that $G \cong G'$

G	G'
2	2
1	2^0
0	2^{-3}
-1	
-2	

$f(m) = 2^m$

1. one-one :- Let $f(m) = f(n)$

$2^m = 2^n$

$\Rightarrow m = n$

① onto :- $\forall\ 2^n \in G'$ $m \in G$
st $f(m) = 2^n$

③ Homomorphism

$$f(m+n) = f(m)\cdot f(n)$$

$$LHS = f(m+n) = 2^{m+n}$$
$$= 2^m \cdot 2^n$$
$$= f(m)\cdot f(n)$$
$$= RHS$$

⇒ Prove that two cyclic group of equal order are isomorphic

Let $f : G \to G'$ st
$f(a^r) = b^r$

$$a \to b,\quad a^2 \to b^2,\quad a^3 \to b^3,\quad \ldots,\quad a^n \to b^n$$

① f is one-one

let $f(a^r) = f(a^s)$
$\Rightarrow a^r = a^s$
$f(a^r) = f(a^s)$
$\Rightarrow b^r = b^s$
$r = s$
$a^r = a^s$ //

② f is onto
$\forall\ b^r \in G'$ $a^r \in G$
st $f(a^r) = b^r$ or [$o(G) = o(G') = n$
Also f is one-one
∴ f is onto]

or $a \neq b \Rightarrow f(a) \neq f(b)$

③ f is Homomorphism

$$f(a^r \cdot a^s) = f(a^r)\cdot f(a^s)$$
$$LHS \Rightarrow f(a^r \cdot a^s) = f(a^{r+s})$$
$$b^{r+s} = b^r \cdot b^s = f(a^r)\cdot f(a^s)$$

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PERMUTATION :-

A one-one mapping of a finite set S onto itself is called a permutation

$$\boxed{\text{Total no. of permutation of n symbol} = n!}$$

⇒ $S_n = n!$

where S_n is set of all permutation in n symbol

example

$S = \{a, b, c\}$

① $\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$ ② $\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$

③ $\begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$ ④ $\begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$

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⑤ $\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$ ⑥ $\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$

⇒ General form :-

$S = \{a_1, a_2, \ldots a_n\}$

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ f(a_1) & f(a_2) & \ldots & f(a_n) \end{pmatrix}$$

Composition of Permutation

$f = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}, g = \begin{pmatrix} b_1 & b_2 & \ldots & b_n \\ c_1 & c_2 & \ldots & c_n \end{pmatrix}$

$fg = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ c_1 & c_2 & \ldots & c_n \end{pmatrix}$

⇒ for identity we check ~~jitha~~ jisme row same hogi wo identity hoga

⇒ taking from ex:-
calculating inverse
let $g = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$, $I = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$

$g \cdot g^{-1} = I$

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$$\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$$

$\begin{pmatrix} a & b & c \\ \cancel{x} & z & \cancel{y} \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \Rightarrow x = a$
$z = b, y = c$

$g^{-1} = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$

Q Calculate inverse

$\begin{pmatrix} 1, & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$

let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$f \cdot f^{-1} = I$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & D \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & c & D & b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$a = 1, c = 2, D = 3, b = 4$

$$f^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

Cyclic permutation

A permutation which replaces object cyclically is called a cyclic permutation or circular permutation.

The number of distinct objects permuted by a cyclic permutation is called length of the cycle.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 5 & 6 \end{pmatrix} = (1, 2\ 4\ 3)$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 6 & 5 & 1 \end{pmatrix} = (1\ 2\ 3\ 4\ 6)$$

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix} = (1,2)(4\ 3)$$

Q) $f = (2\ 3\ 4)$

$g = (5\ 3\ 1\ 2)$

$$f = \begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ 3 & 4 & 2 & 1 & 5 & 6 \end{pmatrix}$$

$$g = \begin{pmatrix} 5 & 3 & 1 & 2 & 4 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$$

$$fg = \begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ 1 & 4 & 5 & 2 & 3 & 6 \end{pmatrix}$$

$$fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 3 & 6 \end{pmatrix} = (1,2)(3\ 4\ 5)$$

$$f \cdot f^{-1} = I$$

$$\boxed{\begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ 3 & 4 & 2 & 1 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}}$$

$$\begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ 3 & 4 & 2 & 1 & 5 & 6 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ b & c & a & d & e & f \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \end{pmatrix}$$

$$f^{-1} = \begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 \\ 4 & 2 & 3 & 1 & 5 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{pmatrix}$$

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TRANSPOSITION :-

A cycle of length 2 is called Transposition

INVERSION :-

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$$P = \begin{pmatrix} 1 & 2 & 3 \cdots & j \cdots & k \cdots & n \\ a_1 & a_2 & a_3 \cdots & a_j \cdots & a_k \cdots & a_n \end{pmatrix}$$

If $(j-k)$ and $(a_j - a_k)$ are of the same sign then (j, k) is called regular otherwise irregular.

The no. of irregular pairs in a permutation are called inversions.

even & odd permutation

If no. of inversion = even no. then even permutation &
if no. of inversion = odd no. then odd permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1, 2) \ (3\ 4)$$ these are transposition

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}$$
$= (1\ 2\ 3)\ (4\ 5)\ (6)$
$\Rightarrow (1\ 2)\ (1\ 3)\ (4\ 5)$
these are transposition

Checking regular and irregular for f

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Pairs are as follows :-
(1 2) (1 3) (1 4)
(2 3) (2 4)
(3 4)

(1 2) is irregular
(1 3) is regular
(1 4) is regular
(2 3) is irregular
(2 4) is regular
(3 4) is irregular.

Since there are 4 irregular pairs so, permutation is even

Group Automorphism

An isomorphism of a group onto itself is called an automorphism.

Ques) $f : R^+ \to R^+$ defined by $f(x) = x^2$ $\forall x \in R^+$ $(R^+, \cdot)$

PT :- F is an automorphism

Soln ① one one :-

$f(a) = f(b) \Rightarrow a = b$

$\because f(a) = f(b)$

$a^2 = b^2$

$\Rightarrow a = b$ Hence it is one one

(2) onto :-

$\forall x \in R^+ \; \exists \sqrt{x} \in R^{-1}$ st

$f(\sqrt{x}) = x$ or $f(x) = x^2$

Hence it is onto

(3) Homomorphism

$f(a.b) = f(a).f(b)$

$LHS = f(a.b) = a^2 b^2$

$RHS = f(a).f(b) = a^2 b^2$

Hence it is Homomorphism

Hence it is automorphism.

THEORAM

Let G be group & f is an automorphism of G. If N is a normal subgroup of G then

PT f(N) is a normal subgroup

Proof :-

(1) To prove f(N) is a subgroup

Let $a', b' \in f(N)$

$\Rightarrow a'(b')^{-1} \in f(N)$

$a', b' \in f(N) \; \exists \; a, b \in N$ st

$a' = f(a)$ & $b' = f(b)$ [By onto]

$\Rightarrow ab^{-1} \in N \qquad \Rightarrow f(ab^{-1}) \in f(N)$

$\Rightarrow f(a).f(b^{-1}) \in f(N)$ [By Homomorph

$\Rightarrow f(a).[f(b)]^{-1} \in f(N)$

$\Rightarrow a'(b')^{-1} \in f(N)$ Hence proved

(2) To prove f(N) is a Normal subgro

Let $x' \in G$, $h' \in f(N)$

$\Rightarrow x'h'(x')^{-1} \in f(N)$

$\exists \; x \in G, \; h \in N$ st

$x' = f(x), \; h' = f(h)$

$\Rightarrow xhx^{-1} \in N$

$f(xhx^{-1}) \in f(N)$

$f(x).f(h).f(x^{-1}) \in f(N)$

$f(x).f(h).[f(x)]^{-1} \in f(N)$

$x'h'(x')^{-1} \in f(N)$ Hence proved

Theorams of Homomorphism

THEORAM I :- If f is a homomorphism from a group G to G' and if e and e' be their respective identities, then:

(a) $f(e) = e'$ (b) $f(a^{-1}) = [f(a)]^{-1}, a \in G$

Proof = (a) Let $a \in G$, then $ae = a = ea$

$\Rightarrow f(ae) = f(a) = f(ea)$

$\Rightarrow f(a).f(e) = f(a) = f(e)f(a)$ [$\because$ f is homomorphism

$\Rightarrow f(e)$ is the identity in $G' \Rightarrow f(e) = e'$

Therefore the image of the identity of G under the group morphism (homomorphism) f is the identity of G'

(b) Let a^{-1} be the inverse of $a \in G$ then

$a \cdot a^{-1} = e = a^{-1} a \Rightarrow f(aa^{-1}) = f(e) = f(a^{-1} a)$

$\Rightarrow f(a) f(a^{-1}) = f(e) = f(a^{-1}) \cdot f(a)$

$\Rightarrow f(a^{-1}) = [f(a)]^{-1}$

Therefore the image of the inverse of any element of G under f is the inverse of the f-image of a in G

→ THEORAM 2:- If f is a homomorphism of a group G to a group G' then

(a) H is a subgroup of $G \Rightarrow f(H)$ is a subgroup of G

(b) H' is a subgroup of $G' \Rightarrow f^{-1}(H') = \{x \in G / f(x) \in H'\}$ is a subgroup of G

Proof (a) clearly $f(H) \subset G'$ and $f(H) \neq \phi$ because $e \in H \Rightarrow f(e) = e' \in H$ where e' is identity in G'

If $a', b' \in f(H)$ then

$a', b' \in f(H) \Rightarrow$ there exist a, b in H

st $f(a) = a'$ and $f(b) = b'$

$\Rightarrow a'(b')^{-1} = f(a)[f(b)]^{-1}$

$= f(a) f(b^{-1})$ $\because [f(b)]^{-1} = f(b)^{-1}$

$= f(ab^{-1})$ $\because$ [f is homomorphism]

But $a \in H$, $b \in H \Rightarrow ab^{-1} \in H$

$\Rightarrow f(ab^{-1}) \in f(H)$

Thus $a', b' \in f(H) \Rightarrow f(ab^{-1}) = a'(b')^{-1} \in f(H)$

$\therefore f(H)$ is a sub group of G

(b) obviously $f^{-1}(H') \subset G$

and $f^{-1}(H') \neq \phi$ because atleast $e \in f^{-1}(H')$

If $a, b \in f^{-1}(H)$ then

$a, b \in f^{-1}(H) \Rightarrow f(a) = H'$ and $f(b) \in H'$

$\Rightarrow f(a)[f(b)]^{-1} \in H'$ [$\because H$ is a subgroup]

$\Rightarrow f(a) f(b^{-1}) \in H'$

$\Rightarrow f(ab^{-1}) \in H'$ [$\because f$ is homomorphism]

$\Rightarrow ab' \in f^{-1}(H')$

Thus $a, b \in f^{-1}(H') \Rightarrow ab^{-1} \in f^{-1}(H')$

$\therefore f^{-1}(H')$ is a subgroup of G

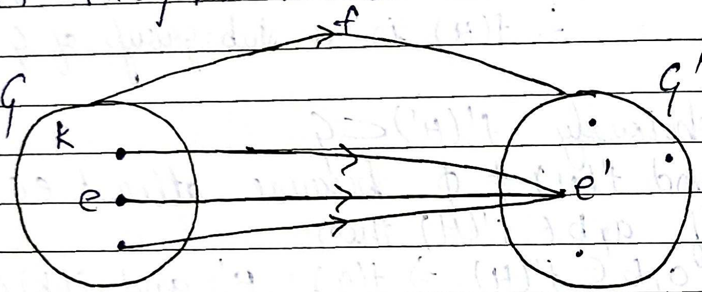
Cor $\Rightarrow$ If f is a homomorphism from a group G to G', then $f(G)$ is a subgroup of G

This can be easily proved by taking $H = G$ in part (a) of the above theorem

Kernal of homomorphism

Let f be a homomorphism of a group G into G' then the set K of all those element of G which are mapped to the identity e' of G' is called the kernal of homomorphism f.
It is denoted by $\ker f$ or $\ker(f)$

$$\ker f = \{x \in G \mid f(x) = e'\}$$



⇒ EXAMPLE 1 :-

The mapping $f: (C_0, \times) \to (R_0, \times)$, $f(z) = |z|$ $\forall z \in C_0$ is a homomorphism of C_0 onto R_0 because for $z_1, z_2 \in C_0$

$$f(z_1 z_2) = |z_1 z_2| = |z_1||z_2|$$
$$\Rightarrow f(z_1 z_2) = f(z_1) \cdot f(z_2)$$

Again $\ker(f) = \{z \in C_0 \mid f(z) = 1\}$
$= \{z \in C_0 \mid |z| = 1\}$

⇒ EXAMPLE 2 :-

$f: R_0 \to R_0$, $f(x) = x^4$, $x \in R_0$ is homomorphism on R_0 because for any $x_1, x_2 \in R_0$

$$f(x_1 x_2) = (x_1 x_2)^4 = x_1^4 \cdot x_2^4$$

$$\Rightarrow f(x_1 x_2) = f(x_1) \cdot f(x_2) \text{ and}$$

$\ker(f) = \{x \in R_0 \mid f(x) = 1\}$
$= \{x \in R_0 \mid x^4 = 1\} = \{1, -1\}$

⇒ EXAMPLE 3 :- The mapping $f: (R, +) \to (C_0, \times)$, $f(x) = e^{ix}$, $\forall x \in R$ is a homomorphism from R to C_0 because $x_1, x_2 \in R$

$$\Rightarrow f(x_1 + x_2) = e^{i(x_1 + x_2)} = e^{ix_1} \cdot e^{ix_2}$$
$$\Rightarrow f(x_1 + x_2) = f(x_1) \cdot f(x_2)$$

again $\ker(f) = \{x \in R \mid f(x) = 1\}$
$= \{x \in R \mid e^{ix} = 1\}$
$= \{x \in R \mid \cos x + i \sin x = 1\}$
$= \{2m\pi \mid m \in Z\} = \{0, \pm 2\pi, \pm 4\pi, \ldots\}$

⇒ EXAMPLE 4 :- If $f: (C, +) \to (R, +)$, $f(x + iy) = x$ then f is a homomorphism from C to R because for any $(x_1 + iy_1), (x_2 + iy_2) \in C$

$$\Rightarrow f[(x_1 + iy_1) + (x_2 + iy_2)] = f[(x_1 + x_2) + i(y_1 + y_2)] = x_1 + x_2$$
$$\Rightarrow f[(x_1 + iy_1) + (x_2 + iy_2)] = f(x_1 + iy_1) + f(x_2 + iy_2)$$

again $\ker(f) = \{(x + iy) \in C \mid f(x + iy) = 0\}$
$= \{(x + iy) \in C \mid x = 0\}$
$=$ the set of imaginary nos.

THEORAM 1:- If f is a homomorphism from a group G to G' with Kernel K then $K \trianglelefteq G$

Proof:- let e and e' be the identities of G and G' respectively then

$Ker(f)$, $K = \{x \in G \mid f(x) = e'\} \subseteq G$

$\therefore f(e) = e' \Rightarrow e \in K \Rightarrow K \neq \phi$

$\therefore$ Let $a, b \in K$, then $f(a) = e'$ and $f(b) = e'$

Again $f(ab^{-1}) = f(a)\, f(b^{-1})$ [$\because f$ is homomorphism]

$\Rightarrow f(ab^{-1}) = f(a)\,[f(b)]^{-1}$

$\Rightarrow f(ab^{-1}) = e'\,(e')^{-1} = e'e' = e'$

$\therefore ab^{-1} \in K$

Thus we see that $a \in K$, $b \in K$, $ab^{-1} \in K$

Therefore the $Ker(f)$ is a subgroup of G

Now prooving $K \trianglelefteq G$:-

let $x \in G$ and $a \in K$

then $f(xax^{-1}) = f(x)\, f(a)\, f(x^{-1})$

$f(xax^{-1}) = f(x)\, e'\, [f(x)]^{-1}$ [$\because a \in K \Rightarrow f(a) = e'$]

$f(xax^{-1}) = f(x)\,[f(x)]^{-1}$

$f(xax^{-1}) = e'$

$\therefore x \in G, a \in K \Rightarrow xax^{-1} \in K$

therefore $K \trianglelefteq G$

THEORAM 2:- Every homomorphic image of a cyclic group is cyclic but not conversely

Proof:-

Let f be a homomorphism of a cyclic group $G = [a]$ to a group G'

$f(G)$ is subgroup of G'

$f(G)$ is cyclic:-

Let $x \in f(G)$, then $x = f(a^n)$ where $n \in Z$

Again $f(a^n) = f(aaaa \ldots n \text{ times})$

$f(a^n) = f(a)\, f(a)\, f(a) \ldots n$ times

$f(a^n) = [f(a)]^n$

which shows that every element of $f(G)$ is some integral power of $f(a)$ i.e

$f(G) = [f(a)]$

Hence $f(G)$ is also a cyclic group

Conversely:- the converse is not necessarily true as can be seen from the given example

$$f: S_3 \longrightarrow \{1, -1\},\ f(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is even} \\ -1 & \text{if } \alpha \text{ is odd} \end{cases}$$

is a homomorphism of S_3 onto $\{1, -1\}$

Moreover the multiplicative group $f(S_3) = \{1, -1\}$ is abelian cyclic but S_3 is neither abelian nor cyclic

THEORAM 3 :- Every group is homomorphic to its quotient group.

Proof :- Let N be a normal subgroup of group G

consider a mapping P from G to G/N defined as :

$P: G \to G/N$, $P(x) = Nx \quad \forall x \in G$

we see that $Na \in G/N$, $\exists\, a \in G$ st $P(a) = Na$

$\therefore$ P is onto

again for any $a, b \in G$

$P(ab) = Nab = NaNb = P(a)P(b)$

$\therefore$ P is epimorphism of G onto G/N

Cor :- If P is homomorphism of G onto G/N defined as above, then $\ker P = N$

Proof :- $P: G \to G/N$, $P(x) = Nx \quad \forall x \in G$ is a homomorphism of G onto G/N

Let $\ker(P) = K$ then

$K = \{x \in G \mid P(x) = N\}$ [N is the identity in G/N]

$K = N$:

If $x \in K \Rightarrow Px = N$

$\Rightarrow Nx = N \Rightarrow x \in N$

$\therefore$ $K \subset N$ —— (1)

again if $x \in N \Rightarrow P(x) = Nx$

$\Rightarrow P(x) = N$ $\quad$ $\because$ [$x \in N \Rightarrow Nx = N$]

$\Rightarrow x \in K$ $\Rightarrow$ $N \subset K$ —— (2)

from (1) & (2) eqn :-

$K = N$

THEORAM 4 :- FUNDAMENTAL THEORAM ON HOMOMORPHISM

Every homomorphic image of a group G is isomorphic to some quotient group of G

Proof :- Let G' be the homomorphic image of group G and f be the corresponding onto homomorphism from G onto G'

If K is the Kernal of f then $K \trianglelefteq G$

Hence G/K is a quotient group of G

To prove that $G/K \cong G'$

Define a map ϕ from G/K to G' as follows :-

$\phi: G/K \to G'$, $\phi(Kx) = f(x)$, $x \in G$

ϕ is well defined

i.e $Ka = Kb \Rightarrow \phi(Ka) = \phi(Kb)$ $\quad a, b \in G$

we see that $Ka = Kb \Rightarrow ab^{-1} \in K$

$\Rightarrow f(ab^{-1}) = e'$ [e' is the identity in G']

$\Rightarrow f(a)\, f(b^{-1}) = e' \Rightarrow f(a)\,[f(b)]^{-1} = e'$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \phi(Ka) = \phi(Kb)$

$\therefore$ ϕ is well defined mapping

again $\phi(Ka) = \phi(Kb) \Rightarrow f(a) = f(b)$

$\Rightarrow f(a)[f(b)]^{-1} = e' \Rightarrow f(b)[f(b)]^{-1} = e'$

$\Rightarrow f(a)\, f(b^{-1}) = e'$

$\Rightarrow f(ab^{-1}) = e' \Rightarrow ab^{-1} \in K$ [$\because K$ is Kernal of f]

$\Rightarrow Ka = Kb$ [$\because Ka = Kb \Leftrightarrow ab^{-1} \in K$]

$\therefore$ ϕ is one one

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$\Rightarrow \phi$ is onto :- Lastly if $a \in G'$ then $a \in G$ st $f(a) = a'$ [$\therefore f$ is onto]

Hence $Ka \in G/K$ such that $\phi(Ka) = f(a) = a'$

$\therefore \phi$ is onto

$\Rightarrow \phi$ is homomorphism :-

Now for any $Ka, Kb \in G/K$

$\phi[Ka\ Kb] \Rightarrow \phi[Kab] \Rightarrow f(ab)$

$\Rightarrow f(a)\, f(b) \Rightarrow \phi(Ka)\, \phi(Kb)$

$\therefore \phi$ is a homomorphism from G/K to G

therefore ϕ is an isomorphism from G/K to G' Hence $G/K \cong G'$

THEORAMS ON ISOMORPHISM

▶ THEORAM 1 :- A homomorphism f defined from a group G onto G' is an isomorphism iff $\ker(f) = \{e\}$

Proof :- suppose f is an isomorphism of G onto G' and K is the kernal of f & $a \in G$ then

$a \in K \Rightarrow f(a) = e'$ [e' is the identity of G']

$\Rightarrow f(a) = f(e)$ [$\therefore f(e) = e'$]

$\Rightarrow a = e$ [$\therefore f$ is one one]

This shows that K contains only the identity e i.e $\therefore K = \{e\}$

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conversely :- suppose that $K = \{e\}$

Let $a, b \in G$ then

$\Rightarrow f(a) = f(b) \Rightarrow$

$\Rightarrow f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$

$\Rightarrow f(a)\, f(b^{-1}) = e'$

$\Rightarrow f(ab^{-1}) = e'$

$\Rightarrow ab^{-1} \in K$ [$\therefore K = \{e\}$]

$\Rightarrow ab^{-1} = e$

$\therefore f$ is bijective homomorphism

Hence it is an isomorphism

▶ THEORAM 2 :- The relation of isomorphism in the set of all group is an equivalence relation

Proof :- ① Reflexive : for any group G the identity mapping I_G defined by $I_G(x) = x$ is an isomorphism because I_G is an abelian and for any $a, b \in G$, $I_G(ab) = ab = I_G(a)\, I_G(b)$

$\Rightarrow G \cong G \Rightarrow$ relation is reflexive

② Symmetric :- Let G and G' be two groups such that $G \cong G'$ and let f be the corresponding isomorphism since by definition f is a bijection so its inverse $f^{-1}: G' \to G$ exist and it is also a bijection further if $a, b \in G$ and $a', b' \in G'$ such that $f(a) = a'$ and $f(b) = b'$ then

$a = f'(a')$ and $b = f^{-1}(b')$ — (1)

then $f'(a'b') = f'[f(a)\, f(b)]$ [by eqn (1)]

$= f^{-1}[f(ab)]$

$= ab$

$= f^{-1}(a')\, f^{-1}(b')$

Therefore f^{-1} is an isomorphism from G' to G

Hence $G' \cong G$

$\therefore$ $G \cong G' \Rightarrow G' \cong G$

therefore the relation is symmetric

(3) Transitive :- let G, G' and G'' be three groups st $G \cong G'$ and $G' \cong G''$

Also let f and g be their respective isomorphism. Since by definition f and g are bijections so

$g \circ f : G \to G''$ is also bijection

$g \circ f$ is homomorphism of G to G''

Hence $g \circ f$ is an isomorphism from G to G'' $\therefore$ $G \cong G''$

therefore the relation is transitive

from the above discussion the relation of isomorphism '$\cong$' is an equivalence relation

THEORAM 3:- Cayle's THEORAM

every group is isomorphic to some permutation group $G \cong P_A$

Proof :- let G be a group corresponding to every a in G we define a map f_a as follows :-

$f_a(x) = ax \quad x \in G$

$\therefore$ $a \in G,\ x \in G \Rightarrow ax \in G$

$\therefore$ $f_a : G \to G$

further for any $x, y \in G$

$f_a(x) = f_a(y) \Rightarrow ax = ay$

$\Rightarrow x = y$ [by canclation law in G]

$\therefore$ f is one one

and for every $x \in G$ there exist $a^{-1}x \in G$ st

$f_a(a^{-1}x) = a(a^{-1}x) = (aa^{-1})x = x$

$\therefore$ f is onto

As such f_a is one one mapping of onto G itself, Hence f_a is permutation of G

let $G' = \{f_a \mid a \in G\}$

clearly $G' \subset S_G$ [S_G is all permutation of G]

let us now considered the mapping f from G to S_G defined by

$\phi : G \to S_G \quad \phi(x) = f_x \quad \forall x \in G$

Now for any $x, y \in G$

$\phi(xy) = f_{xy} = f_x f_y = f(x) f(y)$

$\therefore$ ϕ is a homomorphism from a group

G onto S_G
consequently $f(G) = G$ is a subgroup of the permutation group S_G and ϕ is an epimorphism from G onto G'

Also for any $a, b \in G$
$\Rightarrow \phi(a) = \phi(b) \rightarrow f_a = f_b$
$\Rightarrow f_a(x) = f_b(x) \quad x \in G$
$\Rightarrow ax = bx \Rightarrow a = b$
$\therefore \phi$ is one one
Hence ϕ is an isomorphism from a group G onto permutation group G'
consequently $G \cong G'$

Multiple Integration

Double Integration

Q1) $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

Sol^n $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}$ $\quad \because \int \frac{1}{a^2+y^2} = \frac{1}{a}\tan^{-1}\frac{y}{a}$

$\Rightarrow \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}}$

$\Rightarrow \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}1\right] \Rightarrow \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}dx$

$\Rightarrow \frac{\pi}{4}\left[\log(x+\sqrt{1+x^2})\right]_0^1$

$\Rightarrow \frac{\pi}{4}\left[\left[\log(1+\sqrt{1+1^2})\right] - \left[\log(0+\sqrt{1+0})\right]\right]$

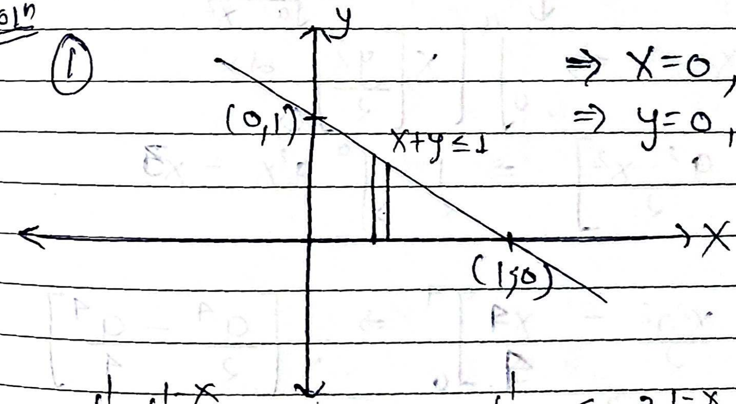
$\Rightarrow \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log 1\right]$

$\Rightarrow \frac{\pi}{4}\log(1+\sqrt{2})$ //

(k) Evaluate $\iint xy\,dy\,dx$ where the following region of integration is
(i) $x+y \leq 1$ in +ve quadrant
(ii) $x^2+y^2 = a^2$ in the +ve quadrant

Sol^n

(1)



$\Rightarrow x=0,\ x=1$
$\Rightarrow y=0,\ y=1-x$

$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dy\,dx \Rightarrow \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x}dx$

$\Rightarrow \int_0^1 x\left[\frac{(1-x)^2}{2}\right] \Rightarrow \frac{1}{2}\int_0^1 x(1-2x+x^2)\,dx$

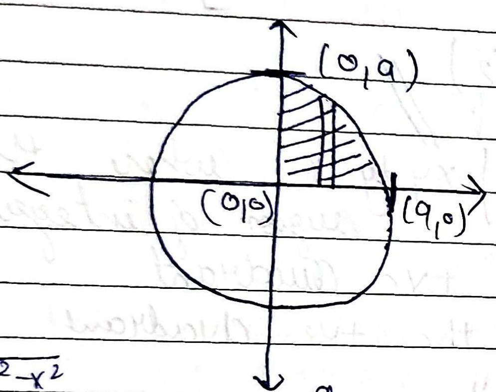
$\Rightarrow \frac{1}{2}\int_0^1 (x-2x^2+x^3)\,dx$

$\Rightarrow \frac{1}{2}\left[\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right]_0^1$

$\Rightarrow \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]$

$\Rightarrow \frac{1}{2}\left[\frac{6-8+3}{12}\right] \Rightarrow \frac{1}{24}$ //

(2)



$\Rightarrow x=0,\ x=a$

$\Rightarrow y=0,\ y=\sqrt{a^2-x^2}$

$\Rightarrow \int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dx \Rightarrow \int_0^a \left[x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}\right] dx$

$\Rightarrow \int_0^a x\left[\frac{a^2-x^2}{2}\right] \Rightarrow \frac{1}{2}\int_0^a a^2x - x^3$

$\Rightarrow \frac{1}{2}\left[\frac{x^2a^2}{2}-\frac{x^4}{4}\right]_0^a \Rightarrow \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]$

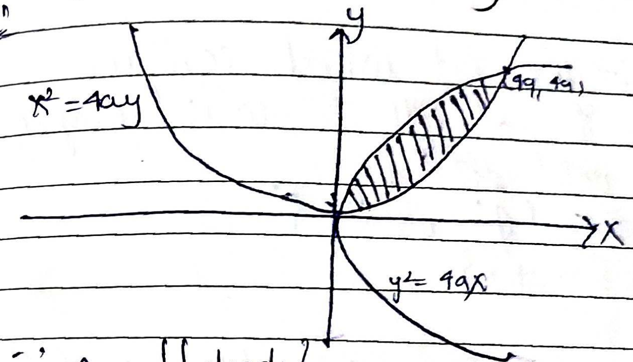
$\Rightarrow \frac{1}{2}\left[\frac{2a^4-a^4}{4}\right] \Rightarrow \frac{a^4}{8}$ //

Area by double integration

$\Rightarrow \boxed{A=\iint dx\,dy}$

Q1) find the area of region by double integral
$y^2=4ax$ & $x^2=4ay$

Sol^n



$\{\because A=\iint dx\,dy\}$

$\Rightarrow A=\int_{x=0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx \Rightarrow \int_0^{4a}\left[y\right]_{x^2/4a}^{2\sqrt{ax}} dx$

$\Rightarrow \int_0^{4a}\left[2\sqrt{a}\sqrt{x}-\frac{x^2}{4a}\right]dx \Rightarrow \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2}-\frac{x^3}{12a}\right]_0^{4a}$

$\Rightarrow 2\sqrt{a}\times\frac{2}{3}(4a)^{3/2}-\frac{(4a)^3}{12a}$

$\Rightarrow 2\sqrt{a}\times\frac{2}{3}\times 8a^{3/2}-\frac{64a^3}{12a}$

$\Rightarrow \frac{32a^2}{3}-\frac{\cancel{64}\,16a^2}{\cancel{12}\,3} \Rightarrow \frac{32a^2}{3}-\frac{16a^2}{3}$

$\Rightarrow \frac{16a^2}{3}$ //